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A discrete Garnier type system from symmetry reduction on the lattice

Anastasios Tongas[1] **and Frank Nijhoff**[2,3]

[1] Department of Mathematics, University of Patras, 26 500 Patras, Greece
[2] Institute for Advanced Study, La Trobe University, Bundoora, Melbourne, Australia

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Abstract

A symmetry reduction of the lattice modified Boussinesq system is studied. The full group of Lie point symmetries of the relevant system is retrieved and certain group invariant solutions are considered by using an accessional generalized symmetry. It is demonstrated that the symmetry reduction leads to a coupled set of second-order nonlinear non-autonomous ordinary difference equations involving six free parameters, generalizing to higher order some of the known discrete analogues of the Painlevé VI equation. The corresponding isomonodromic deformation problem is constructed through the symmetry reduction as well.

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1. Introduction

To a mathematician and a physicist of the time, the year 1905 was certainly eventful. They may have read an article by Fuchs introducing a parameter family of nonlinear, second-order ordinary differential equations, which are widely known nowadays as the Painlevé VI (PVI) equation. Most likely, they may have read the seminal article by Einstein introducing his special theory of relativity [1], setting the key ideas for an entirely new theory of spacetime and gravitation proposed 11 years later, the general theory of relativity. The fact that the equation proposed by Fuchs in 1905 appears systematically in the reduction of the Einstein's field equations of general relativity gives to this centennial a much wider significance.

Fuchs introduced PVI equation in his study on the isomonodromic deformations of a linear, second-order ordinary differential equation of the (Lazarus) Fuchian type with four essential singular points [2]. Contemporaneously, Garnier continued Fuchs' original study, generalizing his results to higher order systems of differential equations and which can be interpreted as higher order analogues of the Painlevé VI equation [3]. The Garnier and the

[3] On leave from Department of Applied Mathematics, University of Leeds, Leeds LS2 9JT, UK.

corresponding Schlesinger systems [4] were systematically studied during the 1980s by the Tokyo school [5, 6] and have recently found a revival of interest.

Discrete Painlevé equations emerged more than a decade ago as difference analogues of the Painlevé transcendental ordinary differential equations. It is firmly believed that discrete Painlevé equations share many of the intriguing properties with the corresponding continuous ones. The first examples of such equations arose from studies on semi-classical orthogonal polynomials [7, 8], random matrix models [9, 10] and integrable lattice systems [11], as discrete analogues of the Painlevé I and II. Subsequently, other examples were found by employing, notably, the very successful method of singularity confinement [12, 13] and the Bäcklund transformations of the continuous Painlevé equations [14]. Since then, the subject developed very rapidly (see [15] for a review) culminating in the work by Sakai [16] who introduced a second-order difference equation with coefficients depending on the discrete variable through elliptic functions. Sakai's equation is regarded as a master discrete Painlevé equation since all other discrete, as well as continuous, Painlevé equations can be obtained from it by degenerations. More recently, Sakai introduced in [17] a q-difference analogue of the Garnier system, by setting up the corresponding q-isomonodromic deformation problem.

Along the same lines of research, a coupled set of second-order ordinary difference equations (OΔEs) was introduced in [18], which can be regarded as the discrete version of a certain system of equations belonging to the Garnier system. The discrete system considered in [18] was the modified lattice Korteweg–de Vries (KdV) system embedded in a three-dimensional lattice and the relevant OΔEs were obtained by employing a compatible three-dimensional symmetry reduction.

In the present work, we investigate a lattice analogue of the (modified) Boussinesq (mBSQ) system and the admitted symmetry group. By applying Lie group techniques to the lattice equations, we consider certain classes of symmetric solutions of the mBSQ system. Special cases of the relevant symmetry constraints were derived in [19]. Here we present in detail the full-parameter case together with its explicit symmetry reduction. It is shown that the reduced system is the following non-autonomous coupled set of second-order OΔEs:

$$\beta_n^1 - n\xi_n = \frac{x_n(r^2y_n-1)\left(\beta_{n+1}^1-(n+1)\xi_{n+1}\right)}{(rx_n-1)(ry_n-x_n)} + \frac{x_n\left(\beta_{n+1}^2-(n+1)\zeta_{n+1}-3c\right)}{ry_n-x_n}, \tag{1a}$$

$$\beta_n^2 - n\zeta_n = \frac{\beta_{n+1}^1-(n+1)\xi_{n+1}}{1-rx_n} + \frac{r\left(\beta_{n+1}^2-(n+1)\zeta_{n+1}\right)}{r-y_n}. \tag{1b}$$

In the above system, $n \in \mathbb{Z}$ is the independent variable and (x_n, y_n) are the dependent variables with values in the complex numbers. The auxiliary quantities $\xi_n, \zeta_n, \beta_n^1, \beta_n^2$ are given by

$$\xi_n = 3\left(1 + \frac{x_n}{y_n}\frac{ry_{n-1}-x_{n-1}}{rx_{n-1}-1} + x_n\frac{r-y_{n-1}}{rx_{n-1}-1}\right)^{-1}, \tag{2a}$$

$$\zeta_n = 3\left(1 + \frac{1}{x_n}\frac{rx_{n-1}-1}{r-y_{n-1}} + \frac{1}{y_n}\frac{ry_{n-1}-x_{n-1}}{r-y_{n-1}}\right)^{-1}, \tag{2b}$$

$$\beta_n^1 = \lambda_1 + \lambda_3\omega^{n+c} + \lambda_4\omega^{2(n+c)} + n + c, \tag{2c}$$

$$\beta_n^2 = -\lambda_2 + \lambda_3\omega^{n+c+1} + \lambda_4\omega^{2(n+c+1)} + n + c, \tag{2d}$$

respectively, where $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, r, c)$ are complex parameters and ω is a primitive cubic root of unity. We call the system (1) discrete Garnier type, as we firmly believe that it constitutes the discrete analogue of the system of higher order ODEs implicit in the original $n = 2$ Garnier

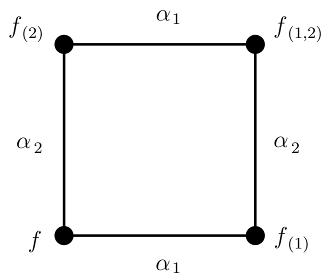

Figure 1. An elementary quadrilateral.

system of ODEs generalizing the PVI equation, cf [3]. However, at this point in time, we have not yet established a direct correspondence with the original system given by Garnier, and hence we call it of 'Garnier type'. These investigations follow closely the approach we have taken in recent works [20, 21], where the so-called *generating* partial differential equation (PDE) of the BSQ hierarchy was derived and shown to constitute a generalization of the Ernst equations for an Einstein–Maxwell–Weyl field. Certain group invariant solutions of the relevant PDEs were investigated and showed that they are built from solutions of a coupled system of second-order nonlinear ODEs involving six free parameters, which was conjectured to be equivalent to the lowest order Garnier system. The results presented in the following constitute the discrete analogue of that procedure.

2. Symmetries and integrability of the discrete Boussinesq system

Whereas lattice equations associated with the KdV equation have been studied extensively over the last decades [22], those associated with the BSQ equation are relatively unknown. In this section, we consider the symmetry group of lattice version of the BSQ and its integrability properties. The relevant lattice equation which was derived in [23] lives on a nine-point stencil. For computational and illustrative reasons, instead of studying the equation on the nine-point stencil, we consider here an equivalent form of the relevant lattice equation by recasting it as a two-component system living on a planar graph with elementary quadrilaterals faces.

For equations of this type, one has fields $f : \mathbb{Z}^2 \to \mathbb{C}^n$ assigned on the vertices at sites (n_1, n_2) which vary by unit steps only and complex lattice parameters α_1, α_2 assigned on the edges of an elementary square (figure 1). The basic building block of such equations consists of a system of algebraic relations of the form

$$\mathcal{B}^{\nu}(f, f_{(1)}, f_{(2)}, f_{(1,2)}; \alpha_1, \alpha_2) = 0, \qquad \nu = 1, \ldots, n, \tag{3}$$

which relate the values of f residing on the four vertices of an elementary quadrilateral. The forward shifted value of a field will be denoted by a subscript inside parentheses with respect to the lattice direction to which the shift operation has been performed and similarly the backward shift is indicated by a minus sign, i.e.

$$f_{(1)} := f(n_1+1, n_2), \qquad f_{(-2)} := f(n_1, n_2-1), \qquad f_{(1,2)} := f(n_1+1, n_2+1). \tag{4}$$

The lattice mBSQ system is a two-component system of equations of the form (3), namely

$$f^1_{(1,2)} = f^2 \frac{\alpha_1 f^1_{(2)} - \alpha_2 f^1_{(1)}}{\alpha_1 f^2_{(1)} - \alpha_2 f^2_{(2)}}, \qquad f^2_{(1,2)} = \frac{f^2}{f^1} \frac{\alpha_1 f^1_{(1)} f^2_{(2)} - \alpha_2 f^1_{(2)} f^2_{(1)}}{\alpha_1 f^2_{(1)} - \alpha_2 f^2_{(2)}}, \tag{5}$$

$f^i : \mathbb{Z}^2 \to \mathbb{CP}^1$, $i = 1, 2$, given first (in a different notation) in [19].

Let us now recall the basic notions of Lie symmetry methods applied to lattice equations of the form (3). A detailed study of symmetry methods applied to algebraic or differential equations can be found in [24, 25]. With minor modifications, such methods can be equally well applied to lattice equations of the form (3).

Let G be a one-parameter group of transformations acting on $U = \mathbb{CP}^1 \times \mathbb{CP}^1$, the domain of the dependent variables of the lattice mBSQ equation, i.e.

$$G : f^i \mapsto \Phi^i(n_1, n_2, f; \varepsilon), \qquad \varepsilon \in \mathbb{C}, \quad i = 1, 2. \tag{6}$$

We denote by $J^{(k)}$ the forward lattice jet space with coordinates $\left(f^i, f^i_J\right)$, where by f^i_J we mean the forward shifted values of f^i, indexed by all unordered (symmetric) multi-indices $J = (j^1, j^2, \ldots j^k)$, $1 \leqslant j^r \leqslant 2$, of order $k = \#J$. Similarly, one can define the backward lattice jet space of order k, denoted by $J^{(-k)}$, or in general the k-order lattice jet space $J^{(k,-k)}$. The prolongation of the group action of G on $J^{(k)}$ is

$$G^{(k)} : \left(f^i, f^i_J\right) \mapsto (\Phi^i(n_1, n_2, f^1; \varepsilon), \Phi^i_J(n_1, n_2, f^i; \varepsilon)), \qquad i = 1, 2, \tag{7}$$

where $\Phi^i_{(1)}(n_1, n_2, f^i; \varepsilon) = \Phi^i\left(n_1 + 1, n_2, f^i_{(1)}; \varepsilon\right)$, $\Phi^i_{(1,2)}(n_1, n_2, f^i; \varepsilon) = \Phi^i\left(n_1 + 1, n_2 + 1, f^i_{(1,2)}; \varepsilon\right)$, etc.

The infinitesimal generator of the group action of G on U is given by the vector field

$$\mathbf{v} = \sum_{i=1}^{2} Q^i(n_1, n_2, f^j)\partial_{f^i}, \qquad \text{where} \qquad Q^i(n_1, n_2, f^j) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Phi^i(n_1, n_2, f^j; \varepsilon)\bigg|_{\varepsilon=0}, \tag{8}$$

$i, j = 1, 2$. There is a one-to-one correspondence between connected groups of transformations and their associated infinitesimal generators since the group action is reconstructed by the flow of the vector field $\mathbf{v}$ by exponentiation

$$\Phi^i(n_1, n_2, f; \varepsilon) = \exp(\varepsilon\mathbf{v}) f^i, \qquad i = 1, 2. \tag{9}$$

The infinitesimal generator of the action of $G^{(k)}$ on $J^{(k)}$ is the associated nth-order forward prolonged vector field

$$\mathbf{v}^{(k)} = \sum_{i=1}^{2} \sum_{\#J=j=0}^{k} Q^i_J(n_1, n_2, f^\ell)\partial_{f^i_J}. \tag{10}$$

By similar considerations, one may define the kth-order backward prolonged vector field $\mathbf{v}^{(-k)}$, and in general the vector field $\mathbf{v}^{(k,-k)}$.

The transformation G is a Lie point symmetry of the lattice equations (3), if it transforms any solution of (3) to another solution of the same equations. Equivalently, G is a symmetry of equations (3), if the equation is not affected by the transformation (7). The infinitesimal criterion for a connected group of transformations G to be a symmetry of equation (3) is

$$\mathbf{v}^{(2)}(\mathcal{B}^\nu(f, f_{(1)}, f_{(2)}, f_{(1,2)}; \alpha_1, \alpha_2)) = 0. \tag{11}$$

Equation (11) should hold for all solutions of equations (3), and thus the latter equations and their consequences should be taken into account. Equation (11) determines the most general infinitesimal Lie point symmetry of the system (3). The resulting set of infinitesimal generators forms a Lie algebra $\mathfrak{g}$ from which the corresponding Lie point symmetry group G can be found by exponentiating the given vector fields.

A (forward) *lattice invariant* for G is a function $I : J^{(k)} \to \mathbb{C}$ which satisfies $I\left(g^{(k)} \cdot \left(f^i, f^i_J\right)\right) = I\left(f^i, f^i_j\right)$ for all $g \in G$ and all $\left(f^i, f^i_j\right) \in J^{(k)}$. For connected groups of transformations, a necessary and sufficient condition for a function $I : J^{(k)} \to \mathbb{C}$ to be invariant under the action of G is the annihilation of I by all prolonged infinitesimal generators, i.e.

$$\mathbf{v}^{(k)}(I) = 0, \tag{12}$$

for all $\mathbf{v} \in \mathfrak{g}$.

We now apply the preceding general framework to the lattice mBSQ system (5). Using the infinitesimal invariance condition (11), we find that the most general group of Lie point symmetry transformations is the following:

$$g^1 : (f^1, f^2) \longmapsto (\mathrm{e}^{\epsilon_1} f^1, f^2), \tag{13a}$$

$$g^2 : (f^1, f^2) \longmapsto (f^1, \mathrm{e}^{\epsilon_2} f^2), \tag{13b}$$

$$g^3 : (f^1, f^2) \longmapsto \left(\mathrm{e}^{\epsilon_3 \omega^{n_1+n_2}} f^1, \mathrm{e}^{-\epsilon_3 \omega^{n_1+n_2+1}} f^2\right), \tag{13c}$$

$$g^4 : (f^1, f^2) \longmapsto \left(\mathrm{e}^{\epsilon_4 \omega^{2(n_1+n_2)}} f^1, \mathrm{e}^{-\epsilon_4 \omega^{2(n_1+n_2+1)}} f^2\right), \tag{13d}$$

where $\epsilon_i \in \mathbb{C}$ and ω is a primitive cubic root of unity. The associated infinitesimal generator of the action of the above symmetry group is given by the following vector field:

$$\mathbf{v} = (\lambda_1 + \lambda_3 \omega^{n_1+n_2} + \lambda_4 \omega^{2(n_1+n_2)}) f^1 \partial_{f^1} + (\lambda_2 - \lambda_3 \omega^{n_1+n_2+1} - \lambda_4 \omega^{2(n_1+n_2+1)}) f^2 \partial_{f^2}. \tag{14}$$

By relaxing the geometrical assumption in which the symmetry characteristics Q^i depend on n_i and f^i and allowing Q^i to be functions defined on $\mathbb{Z}^2 \times J^{(k,-k)}$ for some finite but unspecified $k \in \mathbb{N}, k \geqslant 1$, we arrive naturally at the notion of the generalized Lie symmetries. Symmetry generators of this type cannot be associated with transformation groups acting geometrically on the domain of the dependent variables. Lowest order ($k = 1$) generalized symmetries of the lattice mBSQ system are given by the following three vector fields:

$$\mathbf{w}_i = (\xi^i - 1) f^1 \partial_{f^1} - (\zeta^i - 1) f^2 \partial_{f^2}, \tag{15a}$$

$$\mathbf{z} = (n_1(\xi^1 - 1) + n_2(\xi^2 - 1)) f^1 \partial_{f^1} - (n_1(\zeta^1 - 1) + n_2(\zeta^2 - 1)) f^2 \partial_{f^2}, \tag{15b}$$

where

$$\xi^i = \frac{3 f^1_{(i)} f^2}{f^1_{(i)} f^2 + f^1_{(-i)} f^2_{(i)} + f^1 f^2_{(-i)}}, \qquad \zeta^i = \frac{3 f^1 f^2_{(-i)}}{f^1_{(i)} f^2 + f^1_{(-i)} f^2_{(i)} + f^1 f^2_{(-i)}}, \tag{16}$$

$i = 1, 2$. By exploiting the fact that the commutator of two symmetry generators is again a symmetry generator, one finds from the commutator relations $[\mathbf{z}, \mathbf{w}_i]$ two new symmetry generators with symmetry characteristics defined on $\mathbb{Z}^2 \times J^{(2,-2)}$. In principle, an infinite number of symmetries can be constructed using the commutator relations of the resulting new symmetry generators.

The existence of an infinite number of symmetries is closely related to the integrability properties of the system under consideration. As the definition of the integrability for quadrilateral discrete equations (3), we place the three-dimensional consistency property. It expresses the fact that equations (3) can be embedded consistently in $\mathbb{Z}^3$. More precisely, by the consistency property we mean that the overdetermined system of the equations

$$\mathcal{B}^\nu(f, f_{(i)}, f_{(j)}, f_{(i,j)}; a_i, a_j) = 0, \qquad 1 \leqslant i < j \leqslant 3, \quad \nu = 1, 2, \tag{17}$$

and their shifted versions possesses a non-empty set of solutions. This property can be verified by considering an elementary initial value problem on the three-dimensional cube with initial data assigned on four vertices, not all of them lying on the same face. One such initial configuration is depicted in figure 2 with initial values $(f, f_{(i)})$. In order to prove the three-dimensional consistency, it is sufficient to show that the values at the rest of the vertices

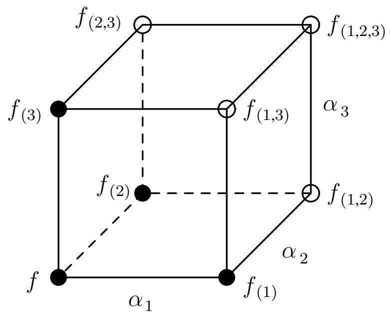

Figure 2. An elementary initial value problem on the cube.

are uniquely determined using the equations on all six faces of the cube. This is achieved as follows: using equations (17) on the three faces adjacent to the vertex with value f, we determine uniquely the values $f_{(i,j)}$ in terms of the initial data. Consecutively, using the equation on each of the remaining three faces, we evaluate $f_{(1,2,3)}$ in three different ways. Consistency means that all these three values are equal in terms of the initial data. After a lengthy but straightforward calculation, we find that the values $f_{(1,2,3)}$ for the discrete mBSQ system are

$$
\begin{aligned}
f^1_{(1,2,3)} &= f^1 \frac{\underset{ijk}{\boldsymbol{\sigma}} a_i a_j f^1_{(k)}\left(a_i f^2_{(i)} - a_j f^2_{(j)}\right)}{\underset{ij}{\boldsymbol{\sigma}} a_i a_j \left(a_i f^1_{(i)} f^2_{(j)} - a_j f^1_{(j)} f^2_{(i)}\right)},\\
f^2_{(1,2,3)} &= f^2 \frac{\underset{ijk}{\boldsymbol{\sigma}} a_i a_j f^2_{(k)}\left(a_i f^1_{(j)} - a_j f^1_{(i)}\right)}{\underset{ij}{\boldsymbol{\sigma}} a_i a_j \left(a_i f^1_{(i)} f^2_{(j)} - a_j f^1_{(j)} f^2_{(i)}\right)},
\end{aligned}
\tag{18}
$$

where $\underset{ijk}{\boldsymbol{\sigma}}$ denotes the cyclic sum over the subscripts $(i, j, k) = (1, 2, 3), (3, 1, 2), (2, 3, 1)$, and similarly the cyclic sum $\underset{ij}{\boldsymbol{\sigma}}$ is over $(i, j) = (1, 2), (2, 3), (3, 1)$. The consistency property follows from the fact that the values $f_{(1,2,3)}$ remain invariant under any permutation of the indices $(1, 2, 3)$.

An immediate consequence of the consistency property is that the mBSQ system can be associated with an auxiliary linear overdetermined system of equations (Lax pair), following a similar approach to the one elaborated in [26, 27]. That linear system, in the mBSQ case, can be constructed by using equations (17) only. Indeed, let us first identify the auxiliary variables $f_3 \in \mathbb{CP}^1 \times \mathbb{CP}^1$, with the ratio of homogeneous variables ψ^i, $i = 0, 1, 2$, as follows:

$$
f^1_{(3)} = \frac{\psi^1}{\psi^0}, \qquad f^2_{(3)} = \frac{\psi^2}{\psi^0}. \tag{19}
$$

Next we insert equations (19) into (17) and set

$$
\psi^0_{(i)} = \alpha_i \frac{f^2_{(i)}}{f^2} \psi^0 - \lambda \frac{1}{f^2} \psi^2, \tag{20}
$$

where $i = 1, 2$ and $\lambda = \alpha_3$. Finally, with these identifications, equations (17) can be written in the following matrix form:

$$
\psi_{(1)} = L^1 \psi, \qquad \psi_{(2)} = L^2 \psi, \tag{21}
$$

where $\psi = (\psi^0, \psi^1, \psi^2)^t$ and the components of the matrices L^1 and L^2 are given by

$$(L^i) = \begin{pmatrix} \alpha_i \frac{f^2_{(i)}}{f^2} & 0 & -\lambda \frac{1}{f^2} \\ -\lambda f^1_{(i)} & \alpha_i & 0 \\ 0 & -\lambda \frac{f^2_{(i)}}{f^1} & \alpha_i \frac{f^1_{(i)}}{f^1} \end{pmatrix}. \tag{22}$$

It is now straightforward to show that the compatibility condition $L^1_{(2)} L^2 = L^2_{(1)} L^1$ of the linear system (21) holds for every value of λ, if and only if equations (5) are satisfied.

3. Symmetry reduction to higher discrete Painlevé equations

In this section, we perform a general symmetry reduction of the lattice mBSQ system to a second-order non-autonomous system of difference equations involving two dependent variables and six free parameters. To be more precise, we search for solutions $f^i = f^i(n_1, n_2; \alpha_1, \alpha_2)$ of the lattice mBSQ system which remain invariant along the orbits of the symmetry generator

$$\mathbf{r} = \mathbf{z} - \mathbf{v}. \tag{23}$$

Since, by definition, the transformation groups generated by the vector fields $\mathbf{z}$, $\mathbf{v}$ act on the domain of the dependent variables f^i only, invariant solutions under $\mathbf{r}$ satisfy in addition to equations (5) the following infinitesimal invariance conditions:

$$\mathbf{r}(f^1) = \mathbf{r}(f^2) = 0. \tag{24}$$

Explicitly, the symmetry constraints (24) take the form

$$n_1 \xi^1 + n_2 \xi^2 = \beta^1, \qquad n_1 \zeta^1 + n_2 \zeta^2 = \beta^2, \tag{25}$$

where

$$\beta^1 = \lambda_1 + \lambda_3 \omega^{n_1+n_2} + \lambda_4 \omega^{2(n_1+n_2)} + n_1 + n_2, \tag{26a}$$

$$\beta^2 = -\lambda_2 + \lambda_3 \omega^{n_1+n_2+1} + \lambda_4 \omega^{2(n_1+n_2+1)} + n_1 + n_2. \tag{26b}$$

This form of the symmetry constraint for the mBSQ system was first proposed in [19] is a more restricted parameter case, which was subsequently used in [28] in a geometric context. The method for obtaining invariant solutions of two-dimensional lattice equations is similar to the direct substitution method for obtaining invariant solutions of PDEs in two independent variables. The aim is to reduce the system of partial difference equations (5), accompanied by the symmetry constraints (25), to a system of ordinary difference equations with respect to one direction of the lattice. Since the discrete mBSQ system is symmetric by interchanging mutually the lattice variables and the corresponding lattice parameters, we choose to eliminate the shifts in the lattice variable n_2. The next step is to determine which will be the new dependent variables of the reduced system. An answer to this problem is provided by the construction of a complete set of joint lattice invariants of the symmetry subgroup $\{g^1, g^2\}$. These are functions on $J^{(2)}$, which remain invariant under the second-order prolongation of the group action of the scaling transformations (13*a*), (13*b*). A complete set of functionally independent invariants of this kind is given by the following six functions:

$$w^i = \frac{f^i_{(2)}}{f^i_{(1)}}, \qquad h^i = \frac{f^i_{(1)}}{f^i}, \qquad g^i = \frac{f^i_{(1,2)}}{f^i_{(1)}}, \tag{27}$$

$i = 1, 2$. The crucial observation is that both the lattice equations and the symmetry constraints can be written in terms of the above invariants and their shifts. Indeed, the lattice mBSQ system can be written in the form

$$h^2 g^1 = \frac{rw^1 - 1}{r - w^2}, \qquad h^2 g^2 = h^1 \frac{rw^2 - w^1}{r - w^2}, \qquad \text{where} \quad r = \frac{\alpha_1}{\alpha_2}. \tag{28}$$

The functions ξ^1, ζ^1, can be written in terms of the invariants w^i and their backward shifts in the direction of the lattice variable n_1 only as follows:

$$\xi^1 = \frac{3}{1 + \frac{w^1}{w^2}\left(\frac{rw^2 - w^1}{rw^1 - 1}\right)_{(-1)} + w^1 \left(\frac{r - w^2}{rw^1 - 1}\right)_{(-1)}}, \tag{29a}$$

$$\zeta^1 = \frac{3}{1 + \frac{1}{w^1}\left(\frac{rw^1 - 1}{r - w^2}\right)_{(-1)} + \frac{1}{w^2}\left(\frac{rw^2 - w^1}{r - w^2}\right)_{(-1)}}. \tag{29b}$$

Similarly, the functions ξ^2, ζ^2 take the form

$$\xi^2 = \frac{3g^1_{(-1)}}{g^1_{(-1)} + \frac{g^2_{(-1)}}{g^1_{(-1,-2)}} + \frac{1}{g^2_{(-1,-2)}}}, \qquad \zeta^2 = \frac{3}{g^2_{(-1,-2)}\left(g^1_{(-1)} + \frac{g^2_{(-1)}}{g^1_{(-1,-2)}} + \frac{1}{g^2_{(-1,-2)}}\right)}. \tag{30}$$

After a lengthy elimination procedure applied to equations (30) and their shifted versions in the direction n_1, using equations (28) and the following identities

$$g^i_{(-1)} = h^i w^i, \qquad g^i_{(-2)} = h^i w^i_{(2)}, \tag{31}$$

$i = 1, 2$, we end up with a system of equations of the form

$$\xi^2 = \frac{w^1(r^2w^2 - 1)\xi^2_{(1)}}{(rw^1 - 1)(rw^2 - w^1)} + \frac{w^1\left(\zeta^2_{(1)} - 3\right)}{rw^2 - w^1}, \tag{32a}$$

$$\zeta^2 = \frac{\xi^2_{(1)}}{1 - rw^1} + \frac{r\zeta^2_{(1)}}{r - w^2}. \tag{32b}$$

Using the symmetry constraints (25) and their shifted versions in the direction n_1, and equations (32), we finally arrive at the reduced system of equations which reads

$$\beta^1 - n_1\xi^1 = \frac{w^1(r^2w^2 - 1)\left(\beta^1_{(1)} - (n_1 + 1)\xi^1_{(1)}\right)}{(rw^1 - 1)(rw^2 - w^1)} + \frac{w^1\left(\beta^2_{(1)} - (n_1 + 1)\zeta^1_{(1)} - 3n_2\right)}{rw^2 - w^1}, \tag{33a}$$

$$\beta^2 - n_1\zeta^1 = \frac{\beta^1_{(1)} - (n_1 + 1)\xi^1_{(1)}}{1 - rw^1} + \frac{r\left(\beta^2_{(1)} - (n_1 + 1)\zeta^1_{(1)}\right)}{r - w^2}, \tag{33b}$$

where ξ^1, ζ^1 and β^1, β^2 are given by equations (29) and (26), respectively. With respect to the lattice variable n_1, equations (33) are exactly the OΔEs quoted in the introduction where they are written in a more customary form.

4. A linear deformation problem

In this section, we consider consistent specifications of the dependence of the main variables f^i and the auxiliary variables ψ^i on the continuous lattice parameters α_i and the spectral parameter λ. These results ultimately rest on the existence of a compatible system of differential-difference equations and partial differential equations which admit a common set of solutions with the original lattice equations under consideration. Consequently, solutions of the reduced system of the difference equations (33) are compatible with special classes of symmetric solutions of the continuous systems.

First, we perform the following gauge transformation on the auxiliary variables ψ of the linear system (21):

$$\psi = \mathcal{G}\varphi, \tag{34}$$

where the components of the matrix $\mathcal{G}$ are given by

$$(\mathcal{G}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & f^1 & 0 \\ 0 & 0 & f^2 \end{pmatrix}. \tag{35}$$

In this gauge, the Lax pair (21) takes the form

$$\varphi_{(1)} = \mathcal{G}_{(1)}^{-1} L^1 \mathcal{G}\varphi = M^1\varphi, \qquad \varphi_{(2)} = \mathcal{G}_{(2)}^{-1} L^2 \mathcal{G}\varphi = M^2\varphi, \tag{36}$$

where the components of the matrices M^i are expressed in terms of the invariants (27). Explicitly, the transformed Lax matrices M^1 and M^2 become

$$(M^i) = \begin{pmatrix} \alpha_i \frac{f^2_{(i)}}{f^2} & 0 & -\lambda \\ -\lambda & \alpha_i \frac{f^1}{f^1_{(i)}} & 0 \\ 0 & -\lambda & \alpha_i \frac{f^1_{(i)}}{f^1}\frac{f^2}{f^2_{(i)}} \end{pmatrix}. \tag{37}$$

The isomonodromic deformation problem for the discrete Garnier type system is based on the following differential-difference system:

$$f^1_{,\alpha_i} = -\frac{n_i}{\alpha_i}(\xi^i - 1) f^1, \qquad f^2_{,\alpha_i} = \frac{n_i}{\alpha_i}(\zeta^i - 1) f^2, \tag{38}$$

where partial differentiation will be denoted by a comma, or by ∂, followed by the variable(s) with respect to which the differentiation has been performed. The system (38) is compatible with the lattice BSQ system, in the sense that the two systems of equations have a non-empty common set of solutions.

In a similar manner as in the lattice BSQ system, equations (38) form their own Lax pair. This Lax pair can be obtained in explicit form by first shifting equations (38) in the third auxiliary direction and using the lattice equations (5) (with one of the lattice directions replaced by the third auxiliary direction) in order to eliminate unwanted double-shifted variables. Thus, we obtain a set of Riccati type equations which can be linearized using the same splitting as in (19), which after employing the same gauge transformation (34) leads to a differential Lax pair. Omitting the details of the derivation here, we present the final result, which consists of the following linear differential relations with respect to the lattice parameters α_i:

$$\varphi_{,\alpha_1} = A^1\varphi, \qquad \varphi_{,\alpha_2} = A^2\varphi, \tag{39}$$

where the components of the matrices A^i read

$$(A^i) = \frac{n_i \vartheta_i}{\alpha_i^3 - \lambda^3} \begin{pmatrix} \alpha_i^2 \frac{f^2_{(-i)}}{f^2} & \lambda^2 & \lambda\alpha_i \frac{f^1_{(-i)}}{f^1} \\ \lambda\alpha_i \frac{f^1_{(i)}}{f^1}\frac{f^2_{(-i)}}{f^2} & \alpha_i^2 \frac{f^1_{(i)}}{f^1} & \lambda^2 \frac{f^1_{(i)}}{f^1}\frac{f^1_{(-i)}}{f^1} \\ \lambda^2 \frac{f^2_{(i)}}{f^2}\frac{f^2_{(-i)}}{f^2} & \lambda\alpha_i \frac{f^2_{(i)}}{f^2} & \alpha_i^2 \frac{f^1_{(-i)}}{f^1}\frac{f^2_{(i)}}{f^2} \end{pmatrix} \tag{40}$$

and the terms ϑ_i are uniquely determined by the relations

$$\operatorname{tr} A^1 = \frac{3n_1\alpha_1^2}{\alpha_1^3 - \lambda^3}, \qquad \operatorname{tr} A^2 = \frac{3n_2\alpha_2^2}{\alpha_2^3 - \lambda^3}. \tag{41}$$

It is now straightforward to show that the equations

$$M^1_{,\alpha_1} + M^1A^1 - A^1_{(1)}M^1 = 0, \qquad M^2_{,\alpha_2} + M^2A^2 - A^2_{(2)}M^2 = 0, \tag{42}$$

resulting from the compatibility conditions $\left(\varphi_{(1)}\right)_{\alpha_1} = (\varphi_{\alpha_1})_{(1)}$ and $(\varphi_{(2)})_{\alpha_2} = (\varphi_{\alpha_2})_{(2)}$ on the linear systems (36), (39), respectively, are satisfied provided equations (38) and their consequences hold.

Let us now restrict our considerations on the dependence of the main and auxiliary variables on the continuous lattice parameters α_1, α_2. The compatibility condition $\varphi_{,\alpha_1\alpha_2} = \varphi_{,\alpha_2\alpha_1}$ on the overdetermined system (39) leads to a system of nonlinear PDEs for the variables f^i and their forward and backward shifts in both directions of the lattice. The resulting system of PDEs and the full algebra of Lie point and generalized symmetries, in connection with the hierarchy of the modified BSQ PDE, are not pertinent in the present discussion. Here, we concentrate on the compatible continuum version of the specific infinitesimal invariance condition (24) on the lattice.

In this direction, a crucial observation is that, in view of the compatible system (38), the symmetry generator $\mathbf{z}$ given by (15*b*) translates to

$$\mathbf{z} = -\left(\alpha_1 f^1_{,\alpha_1} + \alpha_2 f^1_{,\alpha_2}\right)\partial_{f^1} - \left(\alpha_1 f^2_{,\alpha_1} + \alpha_2 f^2_{,\alpha_2}\right)\partial_{f^1}. \tag{43}$$

On the continuum level, the above vector field $\mathbf{z}$ corresponds to the evolutionary vector field of the scaling transformations on f, generated by the vector field

$$\alpha_1\partial_{\alpha_1} + \alpha_2\partial_{\alpha_2}. \tag{44}$$

This observation leads us to conclude that the symmetry generator $\mathbf{z}$ represents the scaling invariance of the compatible system of PDEs. Therefore, in order to construct the compatible linear deformation problem in terms of the continuous lattice parameters, it suffices to lift the symmetry generators $\mathbf{v}$ and (44) on the auxiliary space with coordinates (λ, φ^i) in such a way so as to become symmetry generators of the linear systems (36), (39). Clearly, in the gauge we are working, the symmetry transformations (13*a*), (13*b*) preserve the form of the linear systems (36) and (39). The remaining symmetry transformations, namely (13*c*), (13*d*) and the scaling transformation generated by (44), do not affect the linear systems (36) and (39) by prolonging their group action on the auxiliary space as follows:

$$(\lambda, \alpha_i, f^j, \varphi^k) \mapsto (\mathrm{e}^{\varepsilon_0}\lambda, \mathrm{e}^{\varepsilon_0}\alpha_i, f^j, \mathrm{e}^{\varepsilon_0(n_1+n_2)}\varphi^k), \tag{45a}$$

$$(\lambda, \alpha_i, f^1, f^2, \varphi^0, \varphi^1, \varphi^2) \mapsto (\lambda, \alpha_i, \mathrm{e}^{\varepsilon_3\delta} f^1, \mathrm{e}^{-\varepsilon_3\delta_{(1)}} f^2, \mathrm{e}^{-\varepsilon_3\delta_{(1)}}\varphi^0, \mathrm{e}^{-\varepsilon_3\delta}\varphi^1, \mathrm{e}^{-\varepsilon_3\delta_{(1,1)}}\varphi^2), \tag{45b}$$

$$(\lambda, \alpha_i, f^1, f^2, \varphi^0, \varphi^1, \varphi^2) \mapsto \left(\lambda, \alpha_i, \mathrm{e}^{\varepsilon_4\delta^2} f^1, \mathrm{e}^{-\varepsilon_4\delta^2{}_{(1)}} f^2, \mathrm{e}^{-\varepsilon_4\delta^2{}_{(1)}}\varphi^0, \mathrm{e}^{-\varepsilon_4\delta^2}\varphi^1, \mathrm{e}^{-\varepsilon_4\delta^2{}_{(1,1)}}\varphi^2\right), \tag{45c}$$

where $\delta = \omega^{n_1+n_2}$. From the above group actions, we find that the lifted vector field $\widehat{\mathbf{r}}$ is given by

$$\widehat{\mathbf{r}} = \alpha_1\partial_{\alpha_1} + \alpha_2\partial_{\alpha_2} + \lambda\partial_\lambda + (\gamma_{(1)} - 1)\partial_{\varphi^0} + \gamma\partial_{\varphi^1} + (\gamma_{(1,1)} - 2)\partial_{\varphi^2} - \mathbf{v}, \tag{46}$$

where $\gamma = \lambda_3\omega^{n_1+n_2} + \lambda_4\omega^{2(n_1+n_2)} + n_1 + n_2$. Finally, the associated invariant surface conditions

$$\widehat{\mathbf{r}}(\varphi^i - \varphi^i(\lambda, a_1, a_2)) = 0 \tag{47}$$

deliver the linear deformation problem

$$\lambda\varphi_{,\lambda} + \alpha_1\varphi_{,\alpha_1} + \alpha_2\varphi_{,\alpha_2} = \Gamma\varphi, \tag{48}$$

where the matrix Γ is given as follows:

$$(\Gamma) = \begin{pmatrix} \gamma_{(1)} - 1 & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & \gamma_{(1,1)} - 2 \end{pmatrix}. \tag{49}$$

5. Discussion

Following earlier work [11, 18, 19], we presented a symmetry reduction for the lattice mBSQ system, leading to a coupled set of second-order non-autonomous nonlinear ordinary difference equations. In the continuous case [20, 21], the generating PDE associated with the lattice BSQ system was obtained, which encodes the entire hierarchy of the continuous BSQ system. The corresponding symmetry reduction led to a coupled set of second-order ODEs with six free parameters, generalizing the Painlevé VI equation. The considerations in the present work can be seen as a discrete counterpart of that earlier work.

Whereas the connection between integrable nonlinear evolution equations and Painlevé equations, through similarity reduction, is well known, the connection as established in [18, 21] and in the present work shows that these routes also remain true for higher order systems of the Garnier type, both discrete as well as continuous. The construction employed in [18] is different from the one that we considered here. In the former case, starting from KdV type of lattice systems one obtains a 2×2 matrix isomonodromic deformation system, whereas in the present case starting from the mBSQ type system we obtain a 3×3 matrix Schlesinger type system. It would be of interest to compare these systems and investigate issues of universality between these different Garnier type systems. In fact, a similar programme is underway for the continuous case, cf [26, 27], and it is of interest to see whether on the discrete level these questions become even more pertinent. In particular, issues regarding the *irreducibility* and *transcendentality* of the corresponding general integrals would be of great interest. Finally, it would be interesting to see whether the resulting equations fit in with Sakai's q-analogue of the Garnier system [17].

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